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B. G. GALERKIN'S METHOD IN CALCULUS OF VARIATIONS AND IN THE THEORY OF ELASTICITY

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B. G. GALERKIN'S METHOD IN CALCULUS OF VARIATIONS AND IN THE THEORY OF ELASTICITY

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ABSTRACT. The author reviews the principles of Galerkin's method for the solution of problems of mechanics, and the relation of other approximate methods (Ritz, Rayleigh, Trefftz, Leybenzon) to this method. A list of works in which Galerkin's method is employed is given at the end of the article.

1. A work of B. G. Galerkin entitle "Rods and Plates" (Sterzhni i plastinik) appeared in "Vestnik inzhenerov," No. 19, 1915. In it the author presented a new method for the solution of many problems of structural mechanics, comparing it to the Ritz method. /345*

We permit ourselves to cite an exposition of the nature of this method from this work.

"Let there be a rectangular plate freely supported along the edges which undergoes the load p=f(x,y). Let us set up the equation for the elastic surface:

$$w=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} A_{kn}\sin\frac{k\pi x}{a}\sin\frac{n\pi y}{b}. \tag{b}$$

The element of the coordinates is taken at one of the points, and the axes of the coordinate are directed along the sides of the rectangle.

Equation (b) satisfies the conditions at the ends, because w=0 when x=0, y=0, x=a, y=b; then $\partial^2 w/\partial x^2=0$ when x=0 and x=a, as equally $\partial^2 w/\partial y^2=0$ when y=0 and y=b.

Substituting the expression w into the differential equation of the elastic surface on the bent plate

$$\frac{m^2 Eh^3}{12(m^2-1)}\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=f(x,y), \tag{c}$$

we obtain

$$\frac{m^2 Eh^3\pi^4}{12(m^2-1)}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} A_{kn}\left(\frac{k^2}{a^2}+\frac{n^2}{b^2}\right)^2\sin\frac{k\pi x}{a}\sin\frac{n\pi y}{b}=f(x,y). \tag{d}$$

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* Numbers in the margin indicate pagination in original foreign text.

[1] Leningrad

In the general case, w obviously does not satisfy equation (c) and can be only an approximate solution of the problem. To determine the coefficients of A_{kn} we multiply both halves of equation (d) by $\sin(k\pi x/a)\sin(n\pi y/b)dxdy$ and integrate from 0 to a and from 0 to b. Then we obtain:

$$\frac{m^2 Eh^3 \pi^4}{12(m^2-1)} A_{kn} \frac{ab}{4}\left(\frac{k^2}{a^2}+\frac{n^2}{b^2}\right)^2 = \int_0^a\int_0^b f(x, y)\sin\frac{k\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy = T_{kn}; \qquad \text{(e)}$$

whence

$$A_{kn} = \frac{48(m^2-1)a^3b^3}{m^2 Eh^3}\,\frac{T_{kn}}{(k^2b^2+n^2a^2)^2}. \qquad \text{(f)}$$

Examining equation (b), we note that /346

$$w = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} w_{kn} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} A_{kn}\sin\frac{k\pi x}{a}\sin\frac{n\pi y}{b},$$

where

$$w_{kn} = A_{kn}\sin\frac{k\pi x}{a}\sin\frac{n\pi y}{b},$$

i.e., the elastic surface w consists of many elastic surfaces, as if superimposed.

In this case the right side of equality (e)

$$T_{kn} = \int_0^a\int_0^b f(x, y)\sin\frac{k\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

is the action of external transverse forces when the plate bends along the elastic surface w_{kn} when $A_{kn}=1$." And further:

"Generalizing this technique, we may arrive at the following.

Let us propose a warped form of the plates

$$w = \sum A_n \varphi_n(x, y) \qquad (1)$$

such that each term φ_n satisfies the conditions along the edges; then let us substitute the value of w into the equation

$$\frac{m^2 Eh^3}{12(m^2-1)}\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = f(x, y, w). \qquad (2)$$

Then we obtain:

$$\frac{m^2 Eh^3}{12(m^2-1)} \sum A_n \left(\frac{\partial^4\varphi_n}{\partial x^4} + 2\frac{\partial^4\varphi_n}{\partial x^2 \partial y^2} + \frac{\partial^4\varphi_n}{\partial y^4}\right) = \Phi(x, y).$$

Now, multiplying both halves of the equality by $\varphi_k dxdy$ and integrating over all the area of the plate, we obtain n equations of the form:

$$\frac{m^2 Eh^3}{12(m^2-1)} \sum A_n \iint \left(\frac{\partial\varphi^4_n}{\partial x^4} + 2\frac{\partial^4\varphi_n}{\partial x^2 \partial y^2} + \frac{\partial^4\varphi_n}{\partial y^4}\right)\varphi_k\, dx\, dy = \iint \Phi(x, y)\varphi_k\, dx\, dy. \tag{3}$$

This method may be applied both in problems of elastic oscillations and in problems of the statics of rods and plates."[1]

Thus, as in the Ritz method, expressing the displacement by several "coordinate" functions:

$$w = \sum A_n \varphi_n(x, y)$$

such that each function $\varphi_n(x,y)$ singly satisfies the limiting conditions, academician Galerkin, contrary to Ritz, does not use the potential energy, but equates to zero the force of all external and internal forces on each of the selected virtual displacements for which the variations of $\delta A_n \varphi_n$ are used (the coefficients of A_n may be considered generalized coordinates).

The idea of an approximate application of the element of virtual displacements to the problem of mechanics appeared exceptionally fruitful. /347

The possibility of the application by analogy of the method to problems of variation calculus, mathematical physics and to any functional equation in the hands of the investigator appeared to be a powerful weapon whose value is difficult to overestimate.

In the words of the Englishman Duncan (ref. 11), "it is hardly possible to encounter a problem in mechanics concerning elastic or other deformed bodies to which the Galerkin method might not be successfully applied."

This assertion may as well be extended to problems concerning invariant systems (ref. 9).

Above we noted the difference of the principles on which the Ritz and the Galerkin methods are based. The Ritz method is inapplicable to nonconservative systems, while the Galerkin method, directly applicable to the differential

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[1] The work cited contains many fine examples of the application of this method both to problems of the statics of rods and plates and to the problem of their stability, where the author uses algebraic series as well as trigonometric series.

equation, retains its force. When both methods are applied equivalent results are obtained, such that in this last case the Galerkin method has the advantage of greater simplicity, free from the necessity of obtaining the expression of the functional. (Having specifically this case in mind, Hencky (ref. 2) speaks of "the equally elegant and practically applicable variety of the Ritz method.")

2. Let us examine the problem of the minimum of the integral:

$$J = \iiint\limits_V F(x, y, z, u, u_x', u_y', u_z')\, dx, dy\, dz, \tag{1}$$

where V is the volume limited by the surface Σ, and u is the sought function of the variables x,y,z. Let the values u be given to portions of the surface Σ_1. Let us suppose that F has continuous partial derivatives to the third order inclusive, and that u_x', u_y', u_z' are continuous. The variation of integral (1) is

$$\delta J = \iiint\limits_V \left\{\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x'}\delta u_x' + \frac{\partial F}{\partial u_y'}\delta u_y' + \frac{\partial F}{\partial u_z'}\delta u_z'\right\} dx\, dy\, dz.$$

Transforming the last three terms in the right side according to the Gauss formula, we find:

$$\begin{gathered}\delta J = \iiint\limits_V E\delta u\, dx\, dy\, dz + \\ + \iint\limits_{\Sigma_2} \left\{\frac{\partial F}{\partial u_x'}\cos(nx) + \frac{\partial F}{\partial u_y'}\cos(ny) + \frac{\partial F}{\partial u_z'}\cos(nz)\right\}\delta u\, d\sigma \qquad (\Sigma = \Sigma_1 + \Sigma_2).\end{gathered} \tag{2}$$

Here E designates the left side of the corresponding Euler equation:

$$E = \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x'}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y'}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z'}\right). \tag{3}$$

The required conditions of the minimum are:

$$\delta J = 0. \tag{4}$$

Let us now suppose that the approximate solution of the variational problem can be presented in the form:

$$u(x, y, z) = u_0(x, y, z) + \sum_{i=1}^{n} c_i \varphi_i(x, y, z), \tag{5}$$

where $u_0(x,y,z)$ on the surface satisfies the same conditions as does the sought /348 function u(x,y,z), and c_i designates the undetermined coefficients, while the "coordinate" functions φ_i satisfy the condition

$$\varphi_i = 0 \quad \text{by} \quad \Sigma_1 \qquad (i = 1,\ 2,\ \ldots,\ n). \tag{6}$$

as well as the conditions of differentiability and linear independence.

Then

$$\delta u = \sum_{i=1}^{n} \delta c_i \varphi_i (x,\ y,\ z). \tag{7}$$

Substituting expression (7) into equation (4), we obtain:

$$\sum_{i=1}^{n} \delta c_i \left\{ \iiint_V E\varphi_i \, dx\, dy\, dz + \right. \\ \left. + \iint_{\Sigma_1} \left[\frac{\partial F}{\partial u_x'} \cos(nx) + \frac{\partial F}{\partial u_y'} \cos(ny) + \frac{\partial F}{\partial u_z'} \cos(nz) \right] \varphi_i \, d\sigma \right\} = 0. \tag{8}$$

In view of the arbitrariness of the variation δc_i we arrive at the system of n equations:

$$\iiint_V E\varphi_i \, dx\, dy\, dz + \\ + \iint_{\Sigma_1} \left[\frac{\partial F}{\partial u_x'} \cos(nx) + \frac{\partial F}{\partial u_y'} \cos(ny) + \frac{\partial F}{\partial u_z'} \cos(nz) \right] \varphi_i \, d\sigma = 0 \qquad (i = 1,\ 2,\ \ldots,\ n). \tag{9}$$

Thus, the requirement that the integrals of (9) vanish is essential not for any variation δu, but for n selected in a definite way and having the form $\delta c_i \varphi_i$. In the case of problems of mechanics we arrive at the need that the effect of all the forces vanish in a definite way for the choices n of the virtual displacements.

From the n equations (9) obtained which are linear in the majority of cases in mechanics, generally speaking it is possible to determine the unknown coefficients $c_i (i=1,\ 2,\ \ldots,\ n)$.

We shall designate the equation of the form (9) the Galerkin variational equation.

The Galerkin equation for any variational problem may be written in exactly the same way.

We note in particular that if expression (5) satisfies the limiting conditions for the whole surface Σ, the Galerkin equations have the form:

$$\iiint_V E\varphi_i \, dx\,dy\,dz = 0 \qquad (i = 1, 2, \ldots, n). \tag{9'}$$

However, if the approximate solution of the problem is selected such that expression (5) is an integral of the Euler equation E=0 but does not satisfy the surface conditions, we arrive at the system of equations:

$$\iint_\Sigma \left[\frac{\partial F}{\partial u_x'} \cos(nx) + \frac{\partial F}{\partial u_y'} \cos(ny) + \frac{\partial F}{\partial u_z'} \cos(nz)\right] \varphi_i \, d\sigma = 0 \quad (i = 1, 2, \ldots, n), \tag{9''}$$

which constitutes the contents of the Leybenzon method (ref. 17) and of the equivalent Trefftz method[1], as will be obvious from the following. /349

3. If the variational problem is solved by the Ritz method and by the Galerkin method, identical results will be obtained if one and the same system of "coordinate" functions is selected.

Let us use the Ritz method to seek the integral minimum:

$$J = \iiint_V F(x, y, z, u, u_x', u_y', u_z') \, dx\,dy\,d\tau \tag{1}$$

Let us set

$$u(x, y, z) = u_0(x, y, z) + \sum_{i=1}^{n} c_i \varphi_i(x, y, z), \tag{2}$$

where u_0 and the coordinate functions φ_i are determined as above.

Let the Ritz equations be

$$\frac{\partial}{\partial c_k} \iiint_V F \, d\tau = 0 \qquad (i = 1, 2, \ldots, n). \tag{3}$$

However,

$$\frac{\partial}{\partial c_k} \iiint_V F \, d\tau = \iiint_V \left(\frac{\partial F}{\partial u} \varphi_k + \frac{\partial F}{\partial u_x'} \varphi_{kx}' + \frac{\partial F}{\partial u_y'} \varphi_{ky}' + \frac{\partial F}{\partial u_z'} \varphi_{kz}'\right) d\tau.$$

Transforming the last three terms according to the Gauss formula, we have

$$\iiint_V \left(\frac{\partial F}{\partial u_x'} \frac{\partial \varphi_k}{\partial x} + \frac{\partial F}{\partial u_y'} \frac{\partial \varphi_k}{\partial y} + \frac{\partial F}{\partial u_z'} \frac{\partial \varphi_k}{\partial z}\right) d\tau =$$

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1 Trefftz. Mathematical theory of elasticity (Matematicheskaya teoriya uprugosti). Pages 162-163.

$$= \iint_{\Sigma_2} \left[\frac{\partial F}{\partial u_x'} \cos(nx) + \frac{\partial F}{\partial u_y'} \cos(ny) + \frac{\partial F}{\partial u_z'} \cos(nz) \right] \varphi_k \, d\sigma -$$

$$- \iiint_V \left[\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x'} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y'} \right) + \frac{\partial}{\partial z} \left(\frac{\partial F}{\partial u_z'} \right) \right] \varphi_k \, d\tau .$$

Thus,

$$\frac{\partial}{\partial c_k} \iiint_V F \, d\tau = \iiint_V \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x'} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y'} \right) - \frac{\partial}{\partial z} \left(\frac{\partial F}{\partial u_z'} \right) \right] \varphi_k \, d\tau +$$

$$+ \iint_{\Sigma_2} \left[\frac{\partial F}{\partial u_x'} \cos(nx) + \frac{\partial F}{\partial u_y'} \cos(ny) + \frac{\partial F}{\partial u_z'} \cos(nz) \right] \varphi_k \, d\sigma = 0 ,$$

i.e., we have arrived at the Galerkin variation equation (9) of the previous section[1].

The proven statement (obviously suitable for any variational problem) permits all the results obtained for the Ritz method, concerning the proofs of the existence of a solution and convergence in particular to be almost automatically transferred to the Galerkin method.

Along with this, since the Galerkin method appears to be more practically applicable than the Ritz method, by significantly simplifying the calculations it may completely replace this latter and in all those problems where possible, particularly in problems of mechanics, the application of both methods. /350

4. The Galerkin equations for the approximate solution of the spatial problem of the theory of elasticity can be obtained both by direct application of the element of virtual displacements, and in view of the existence of an elastic potential from the principle of minimum potential energy.

In the case of the functional of the form[2]

$$J(U, V, W) = \iiint_V F(x, y, z, U, U_x', U_y', U_z', V, V_x', \ldots, W, W_x', \ldots) \, dx \, dy \, dz +$$

$$+ \iint_{\Sigma_2} \Phi(u, v, U, V, W) \, d\sigma , \tag{1}$$

where U, V, W are the sought functions, while u and v are parameters determining the position of the points on the surface, having set

$$U = U_0 + \sum_{i=1}^{n} a_i \varphi_i , \qquad V = V_0 + \sum_{i=1}^{n} b_i \psi_i , \qquad W = W_0 + \sum_{i=1}^{n} c_i \chi_i , \tag{2}$$

[1] Incidently. the equivalence of the methods of Leybenzon and Trefftz also follows from this.

[2] Compare Smirnov, Krylov, Kantorovich. Variational Calculus (Variatsionnoye ischisleniye) pp. 88-89.

where U_0, V_0, W_0 take the values given for U, V, W on part of the surface Σ_1, while the coordinate functions $\varphi_i, \psi_i, \chi_i$ vanish for Σ_1, each singly ($\Sigma = \Sigma_1 + \Sigma_2$), similarly to the preceding, it is easy to obtain the Galerkin variational equation.

They have the form (i=1, 2, ..., n):

$$\iiint_V \left\{\frac{\partial F}{\partial U} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial U_x'}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial U_y'}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial U_z'}\right)\right\} \varphi_i \, d\tau +$$
$$+ \iint_{\Sigma_1} \left\{\frac{\partial F}{\partial U_x'} \cos(nx) + \frac{\partial F}{\partial U_y'} \cos(ny) + \frac{\partial F}{\partial U_z'} \cos(nz) + \frac{\partial \Phi}{\partial U}\right\} \varphi_i \, d\sigma = 0; \qquad (3)$$

two more 2n equations for V and W have a similar form.

The full potential energy of the elastic body (expressed in displacments) is a functional of the form (1). Applying to it the Galerkin variational equations (3), we proceed to the system of 3n linear equations for the approximate solution of spatial problems in the theory of elasticity[1].

The same result can be obtained by direct application of the element of virtual displacements, similar to the way in which the author obtained the equation for plates. Using the equations of equilibrium in displacements:

$$(\lambda + \mu)\frac{\partial \theta}{\partial x} + \mu \nabla^2 u + \rho X = 0, \qquad (\lambda + \mu)\frac{\partial \theta}{\partial y} + \mu \nabla^2 v + \rho Y = 0,$$
$$(\lambda + \mu)\frac{\partial \theta}{\partial z} + \mu \nabla^2 w + \rho Z = 0. \qquad (4)$$

The surface conditions

$$\begin{aligned}
F_{nx} &= (\lambda\theta + 2\mu e_{xx})\cos(nx) + \mu e_{xy}\cos(ny) + \mu e_{xz}\cos(nz), \\
F_{ny} &= \mu e_{yx}\cos(nx) + (\lambda\theta + 2\mu e_{yy})\cos(ny) + \mu e_{yz}\cos(nz), \\
F_{nz} &= \mu e_{zx}\cos(nx) + \mu e_{zy}\cos(ny) + (\lambda\theta + 2\mu e_{zz})\cos(nz).
\end{aligned} \qquad (5)$$

Let us decompose the displacements by the coordinate functions: /351

$$\begin{aligned}
u &= u_0(x, y, z) + \sum_{i=1}^{n} a_i \varphi_i(x, y, z), \\
v &= v_0(x, y, z) + \sum_{i=1}^{n} b_i \psi_i(x, y, z), \\
w &= w_0(x, y, z) + \sum_{i=1}^{n} c_i \chi_i(x, y, z),
\end{aligned} \qquad (6)$$

[1] They were similarly obtained by Professor A. I. Lur'ye in his book (ref. 8).

for which u_0, v_0, w_0 correspond with the given values of the displacements on the part of the surface Σ_1 where the displacements are given, while all functions $\varphi_i, \psi_i, \chi_i$ in this part of the surface vanish and in the remaining part are arbitrary, satisfying only the conditions of differentiability and linear independence.

We select the virtual displacements in the form:

$$\varphi_i(x, y, z)\delta a_i, \qquad \psi_i(x, y, z)\delta b_i, \qquad \chi_i(x, y, z)\delta c_i, \tag{7}$$

where $\delta a_i, \delta b_i, \delta c_i$ are variations of the undetermined coefficients in expression (6).

Let us calculate the action of the mass and surface forces on the selected virtual displacements. From the first equation (4)

$$\rho X = -\left[(\lambda + \mu)\frac{\partial\theta}{\partial x} + \mu\nabla^2 u\right].$$

Multiplying by the i-th virtual displacement $\varphi_i(x,y,z,)\delta a_i$ and integrating with respect to volume V, we obtain:

$$\begin{gathered}\iiint_V \rho X\varphi_i(x, y, z)\delta a_i\, dx\, dy\, dz = \\ = -\iiint_V \left[(\lambda + \mu)\frac{\partial\theta}{\partial x} + \mu\nabla^2 u\right]\varphi_i(x, y, z)\delta a_i\, dx\, dy\, dz.\end{gathered} \tag{8}$$

Multiplying both parts of the first of equations (5) by the i-th virtual displacement parallel to the x axis and integrating over the surface, we obtain:

$$\begin{gathered}\iint_{\Sigma_1} F_{nx}\varphi_i(x, y, z)\delta a_i\, d\sigma = \\ = \iint_{\Sigma_1} [(\lambda\theta + 2\mu e_{xx})\cos(nx) + \mu e_{xy}\cos(ny) + \mu e_{xz}\cos(nz)]\varphi_i(x, y, z)\delta a_i\, d\sigma.\end{gathered} \tag{9}$$

Adding equalities (8) and (9), we arrive at the Galerkin equation

$$\begin{gathered}\iiint_V \left[(\lambda + \mu)\frac{\partial\theta}{\partial x} + \mu\nabla^2 u + \rho X\right]\varphi_i\, d\tau = \\ = \iint_{\Sigma_1} [(\lambda\theta + 2\mu e_{xx})\cos(nx) + \mu e_{xy}\cos(ny) + \mu e_{xz}\cos(nz)]\varphi_i\, d\sigma.\end{gathered} \tag{10}$$

The remaining equations may be written comparably. Summing up in the left part the forces of inertia, we obtain the equation of motion of the elastic body.

The stated reasoning can obviously be carried to the case of any orthogonal coordinate system. /352

5. The majority of authors feel that the essential difference between the Galerkin and Ritz methods is that, applying the Ritz method, it is sufficient to satisfy only the geometric limiting conditions, while in applying the Galerkin method, along with the geometric conditions the dynamic conditions must also be satisfied (the vanishing of zero forces or moments at the free ends).

Thus, for example, Biezeno and Grammel (ref. 6) cite the conclusion of the Galerkin equations for the spatial problem in the theory of elasticity and show the equivalence of the results for both methods if only conditions (5) of the preceding section are satisfied.

In another work Grammel (ref. 7), using the "variational principle," cites the problem of eigenvalues to the equation:

$$\int_0^1 \left[\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \lambda\rho y\right]\delta y\,dx + \left\{\frac{\partial F}{\partial y''}\delta y' - \left[\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) - \frac{\partial F}{\partial y'}\right]\delta y\right\}_0^1 = 0. \qquad (1)$$

In the author's opinion, the system of coordinate functions for application of the Galerkin method must also satisfy all dynamic limiting conditions in relation to which the nonintegral terms vanish.

There is no justification for the stated requirement that the dynamic limiting conditions be satisfied by coordinate functions when the Galerkin method is applied. This is so, in the first place, as is shown by the fact that the Galerkin method and the Ritz method are equivalent, as is shown in Section 3. On the other hand, this is so, as was shown in a preceding issue, due to the equations of the theory of elasticity by the approximate application of the element of virtual displacements.

If when solving the problems of mechanics by the Galerkin method we start from the principle of variation, in which the system of coordinate functions does satisfy the dynamic conditions, these conditions are satisfied "automatically" even if we start with the differential equation of equilibrium, then it is evident that we must take into account the action of the surface forces on that part of the border where the displacements are not given.

In equation (1), for example, the dynamic border conditions are satisfied "automatically" when the nonintegral terms present an additional action of the surface forces.

6. The Galerkin method has found numerous applications in the static problems in the theory of elasticity.

Here we may point out one modification of the method: the Kantorovich-

Galerkin method, which has been shown to be extremely convenient in the solution of countless classical problems in the theory of elasticity (Saint Venant torsion and flexure) and in many cases easily reduces to a precise solution. The Kantorovich method[1] is contained basically in the following. /353

It is well-known that the problem of the torsion of a weightless prismatic rod can be reduced to calculation of the variation of the integral (ref. 12):

$$\iint \left\{ \frac{1}{2} \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 \right] - 2\Phi \right\} dx\,dy. \tag{1}$$

According to the Kantorovich variational method, the function Φ will be sought in the form of the product:

$$\Phi(x, y) = f(x, y)\,\varphi(x), \tag{2}$$

where $f(x,y)$ is the arbitrarily chosen function, while $\Phi(x,y)$ is taken on the boundary with the possible exception of the segments parallel to the y axis, the given values, and $\varphi(x)$ is an unknown function.

Substituting Φ into expression (1) according to (2), let us integrate with respect to y, then let us compose the Euler equation for the determination of the function $\varphi(x)$.

In the Galerkin-Kantorovich method they proceed immediately from the differential equation for the determination of the torsion function Φ

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -2 \tag{3}$$

and the boundary condition $\Phi=0$.

Presenting Φ with respect to (2), we select the virtual displacement in the form:

$$\delta\Phi = f(x, y)\,\delta\varphi(x). \tag{4}$$

Substituting the chosen function Φ into equation (3) with respect to (2) and multiplying both sides of the equation by the virtual displacment $\delta\Phi$, let us integrate it with respect to y in the corresponding limits; considering the arbitrariness of $\delta\varphi$, we arrive at the ordinary second-order differential equation relative to the function $\varphi(x)$.

The Prandtl analogy[2] permits us to treat the equation

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[1] See in more detail: Kantorovich and Krylov. Methods for the approximate solution of partial differential equations (Metody priblizhennogo resheniya uravneniy v chastnykh proizvodnykh). pp. 248-257.

[2] Timoshenko. Course in the theory of elasticity (kurs teorii uprugosti) 1914 edition, Part 1, p. 160.

$$\iint_\Sigma \nabla^2 \Phi \delta\Phi \, d\sigma = -2 \iint_\Sigma \delta\Phi \, d\sigma \qquad (5)$$

as the condition of vanishing of the action of all forces applied to the membrane on the chosen virtual displacement.

We note that if Φ is presented not in the form (2), but is expanded with respect to the coordinate functions

$$\Phi(x, y) = \sum_{i=1}^{n} c_i \varphi_i , \qquad (6)$$

where all φ_i are equal to zero on the boundary of the transverse section of the rod, we arrive at the ordinary Galerkin method.

The Galerkin-Kantorovich method and the solution (with the aid of this method) for the problem of the torsion of a trihedral prism were expanded for the first time, as far as we know, by Academician Galerkin at his lectures read in 1937 for a group of his co-workers and graduate students. A precise solution is obtained for the case of an equilateral triangle[1]. /354

This method is applied identically to the problem of flexure.

V. Z. Vlasov (ref. 16) applied it to the computation for rectangular plates.

To illustrate, let us examine the problem of the flexure of a weightless prism whose transverse section is a right isosceles triangle, and the force Q is applied at the center of gravity of the end section and parallel to the x axis. The equations for the ends of the triangle are:

$$x+y=0, \qquad x-y=0, \qquad y=b.$$

The flexure function φ must be determined from the equation[2]

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{\sigma}{1+\sigma}\frac{Qy}{J} - f'(y), \qquad (7)$$

where f(y) is an arbitrary function and the function φ satisfies the boundary condition:

1

This solution is presented by Lur'ye in his "Course on the theory of elasticity" (Kurs teorii uprugosti) (ref. 8).

2

Timoshenko. Course in the theory of elasticity. 1914 edition, Part 1, pp. 175-180.

$$\frac{\partial\varphi}{\partial s}=\left[\frac{Qx^2}{2J}-f(y)\right]\frac{\partial y}{\partial s}. \tag{8}$$

Let us set $f(y)=Qy^2/(2J)$. The on the lateral sides of the triangle $Qx^2/(2J)-f(y)=0$; on the vertical side $y=b$ we have $\partial y/\partial s=0$ and, according to (8), $\partial\varphi/\partial s=0$ eveywhere on the boundary.

On the boundary, let $\varphi=0$.

Let us set

$$\varphi=(x^2-y^2)\psi(y), \qquad \delta\varphi=(x^2-y^2)\delta\psi(y). \tag{9}$$

After substituting the value $f'(y)$ and expression (9) into (7) for φ, we multiply both parts of the equation obtained by $\delta\varphi$ and integrate it with respect to x. In virtue of the arbitrariness of $\delta\varphi$, we finally obtain the differential equation:

$$y^2\psi''(y)+5y\psi'(y)=\frac{5}{4}\frac{1}{1+\sigma}\frac{Qy}{J}. \tag{10}$$

The general solution to equation (10) has the form:

$$\psi=\frac{1}{4}\frac{1}{1+\sigma}\frac{Qy}{J}+C_1+C_2y^{-4}. \tag{11}$$

The arbitrary constants are determined from the conditions $\psi(b)=0$ and the boundedness of the function in the element of the coordinate.

Thus, we obtain:

$$\varphi(x,y)=\frac{1}{4(1+\sigma)}\frac{Q}{J}(y^2-x^2)(b-y). \tag{12}$$

Expression (12) is a precise solution[1] of the problem, since it satisfies equation (7). We will not stop at the solution of terms of the corresponding torsion. A precise solution of the Poisson solution for the observed profile can be taken from the work of B. G. Galerkin[2] "Torsion of a trihedral prism."

A. I. Lur'ye (ref. 36) applied this method to problems of torsion. /355

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[1] Apparently proceeding from different calculations, Galerkin obtained the same solution of his work "Torsion of a prism whose base is a right isosceles triangle" Comptes Rendus de l'Academie des Sciences de Paris, Vol. 180, p. 1825, 1925.

[2] Izvestiya Rossiyskoy Akademii Nauk, Series 6, No. 12, pp.111-118, 1919.

It is especially valuable to apply the Galerkin method to questions of the strength and stability of shells. Calculation of potential energy is a very complicated problem in this case, so that the Galerkin method, immediately becoming the differential equation, fully realizes its advantages.

This method has been used in the calculation of the strength of cylindrical shells on two supports under the action of internal hydrostatic pressure.

We have used the differential equation for the equilibrium of cylindrical shells obtained by Galerkin[1].

The question reduces to the integration of the partial differential equation of the eighth order whose partial derivatives have constant coefficients of the form

$$A\frac{\partial^8\varphi}{\partial\theta^8}+B\frac{\partial^8\varphi}{\partial\theta^6\,\partial\zeta^2}+C\frac{\partial^8\varphi}{\partial\theta^4\,\partial\zeta^4}+D\frac{\partial^8\varphi}{\partial\theta^2\,\partial\zeta^6}+F\frac{\partial^8\varphi}{\partial\zeta^8}+A\frac{\partial^6\varphi}{\partial\theta^6}+M\frac{\partial^6\varphi}{\partial\theta^4\,\partial\zeta^2}+ \\ +N\frac{\partial^6\varphi}{\partial\theta^2\,\partial\zeta^4}+L\frac{\partial^6\varphi}{\partial\zeta^6}+P\frac{\partial^4\varphi}{\partial\theta^2\,\partial\zeta^2}+Q\frac{\partial^4\varphi}{\partial\zeta^4}=\frac{1-\sigma^2}{Ea}p_a\,a. \qquad (13)$$

This method aided Galerkin's success in obtaining a precise solution of this problem[2].

The Galerkin method can apparently be equally successfully applied to the problem of the stability of cylindrical shells (work in this direction is being conducted in the Group in Structural Mechanics of the Research Institute for Hydraulic Engineering, under the direction of Galerkin).

Lur'ye applied the Galerkin method to the problem of the stability of shells in his course on the theory of elasticity.

7. The Galerkin method and its various modifications have found numerous application in problems of the oscillations of elastic and invariable systems.

It is easy to see (p.6) that application of the Ritz method to problems of motion starting from the Hamilton principle leads to results identical to those of the application of the Galerkin method starting from the Lagrange equations (refs. 32,35). Lur'ye and Chermarev (ref. 9) applied the Galerkin method to the problem of forced vibration, where the Ritz method cannot be applied.

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[1] Galerkin, B. G. The theory of elastic cylindrical shells (K teorii uprugoy tsilindricheskoy obolochki) Doklady Akademii Nauk (DAN), 1934.

[2] For another solution see B. G. Galerkin and Ya. I. Perel'man: Stresses and displacements in cylindrical conduits (Napryazheniya i peremeshcheniya v tsilindricheskom truboprovode). Izvestiya Nauchno-issledovatel'skogo instituta gidrotekhniki, No. 27.

In many of his works, Professor Duncan[1] uses the Galerkin method, which he considers to be an approximate application of the Lagrange equations, to solve many problems concerning the oscillations of elastic bodies[2]. He showed the equivalence of the Galerkin and Rayleigh methods when one of the systems of functions is used to determine the frequency of free oscillations. Duncan directs particular attention to the selection of factors by which the differential equations are multiplied, recommending pursuing the fact that the Galerkin equations had a determined physical meaning[3]. /356

The Galerkin method received further development in problems of oscillations in the works of Grammel (ref. 7). Grammel reduces the problem of finding eigenvalues to a homogeneous integral equation which is solved essentially by use of the Galerkin method (although Grammel goes so far as to compare his method to the Galerkin method).

As the author shows, the Grammel method, for various systems of coordinate functions give fewer and consequently more precise upper limits for all eigenvalues (not only for the first) than the ordinary Galerkin method. Following Grammel, Weinel (ref. 18) gives a method for finding not only eigenvalues, but corresponding eigenfunctions with arbitrary degrees of accuracy as well (by use of the method of successive approximations).

A. I. Lenchenko applied the Galerkin method to the problem of the oscillations of arches (ref. 39). Ye. P. Grossman applied the Galerkin method to the integration of equations of flutter (ref. 31). In a recently published work (ref. 38), G. I. Petrov applied the Galerkin method to the problem of the stability of the flow of a viscous fluid, rigorously proving the solution.

Yu. D. Repman (ref. 37) has shown that application of the Galerkin method to the problem of the stability of elastic systems with the incorrect selection of functions (without sufficient consideration for the mechanical properties of the functions and their derivatives) can lead to highly erroneous results if we proceed from the differential equation

$$L[y]=0. \qquad (1)$$

Repman shows the necessity of formal mathematical proof of the method in its general form. At the same time, the method applied to the corresponding

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[1] See References.

[2] Particularly, he solves the problem of the effect of the flexibility of rear support of the surfaces on the frequency of oscillations when the body undergoes torsion. R and M, No. 1849.

[3] B. G. Galerkin also pointed this out presenting a course on his method. See also the work of Yu. D. Repman (ref. 37).

integral equation

$$y - A[y] = 0, \qquad (2)$$

can be easily proved with the arbitrary selection of a completely orthogonal system of functions.

In the work of Hencky, the Galerkin method also found application in the theory of plasticity (ref. 3). D. Yu. Panov applied it to several nonlinear problems in the theory of elasticity (ref. 15).

The method of B. G. Galerkin can also be applied as the method of solving differential and integral equations in general.

Thus, in the work of Jones and Scan (ref. 25) several examples of the solution of differential equations according to the Galerkin method are applied and compared with other methods (Taylor, least squares, "collocations"). The Galerkin method shows itself to be most effective. Unfortunately, there is absolutely no mathematical proof in the work. Error is admitted in the Duncan proof (ref. 11) of the identicalness of the Galerkin method and the method of least squares in the solution of ordinary differential equations when $n \to \infty$.

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